MULTIPARAMETER RESOLVENT TRACE EXPANSION FOR ELLIPTIC BOUNDARY PROBLEMS

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Abstract. We establish multiparameter resolvent trace expansions for elliptic boundary value problems, polyhomogeneous both in the resolvent and the auxiliary parameter. The present analysis is rooted in the joint project with Matthias Lesch on multiparameter resolvent trace expansions on revolution surfaces with applications to regularized sums of zeta-determinants.

1. Introduction and formulation of the main result

Consider a compact manifold M of dimension m with boundary ∂M, equipped with a Hermitian vector bundle (E, h^E) of rank p. We recall the notion of an elliptic boundary problem from Seeley [See69]. Let $A \in \mathrm{Diff}^q(M, E)$ denote a differential operator on M with values in E of q-th order with $pq \in 2\mathbb{N}_0$. The operator A is elliptic if its principal symbol $\sigma(A)(p, \zeta)$ is invertible for $(p, \zeta) \in T^*M \backslash \{0\}$. Assume, A satisfies the Agmon condition in a fixed cone $\Gamma' = \{z \in \mathbb{C} \mid \arg(z) \in (\theta_1, \theta_2)\}$ of the complex plane, i.e. $(\sigma(A) + z^q)$ is invertible for $z^q \in \Gamma'$. Consider a system of differential operators $B = (B_1, .., B_{pq/2})$ on ∂M, such that (A, B) defines an elliptic boundary problem satisfying the Agmon condition on Γ' in the sense of [See69, Def. 1,2].

Under this setup, (A, B) defines a closed unbounded operator A_B on $L^2(M, E)$, obtained as the graph closure of A acting on $u \in C^\infty(M, E)$ satisfying the boundary conditions $Bu = 0$. Moreover, $(A_B + z^q)$ is invertible to $z^q \in \Gamma'$ sufficiently large and the seminal work of Seeley [See69] establishes an expansion of the resolvent $(A_B + z^q)^{-1}$ as $|z| \to \infty$. In the present paper we fix $W \in C^\infty(M, E)$ and a finite collection of scalar smooth potentials $V_1, .., V_n \in C^\infty(M)$, constant along ∂M.

Assume for simplicity that each $V_k(0) = V_k \restriction \partial M > 0$. Set $\Gamma := \{z \in \mathbb{C} \mid z^q \in \Gamma'\}$ and write Γ^n for its n-th Cartesian product. Consider $\lambda = (\lambda_1, ..., \lambda_n) \in \Gamma^n$ and the corresponding the multiparameter family

$$A_B(\lambda) := A_B + \sum_{k=1}^{n} \lambda_k^q V_k + W.$$

For $qN > m$ the N-th power of the resolvent $(A_B(\lambda) + z^q)^{-N}$ is trace class and our main result establishes a multiparameter expansion of the resolvent trace $\mathrm{Tr}(A_B(\lambda) + z^q)^{-N}$, polyhomogeneous in $(z, \lambda) \in \Gamma^{n+1}$.

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Theorem 1.1. *Consider any multiindex $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0$. Fix $N \in \mathbb{N}$ such that $qN > m$. Then there exist $e_i \in C^\infty(M \times (\Gamma^{n+1} \cap \mathbb{S}^n))$, such that*

$$\partial_\lambda^\alpha \partial_z^\beta \operatorname{Tr}(A_B(\lambda) + z^q)^{-N} \sim \sum_{j=0}^\infty e_j \left(\frac{(\lambda, z)}{|(\lambda, z)|} \right) |(\lambda, z)|^{-1-qN-j-|\alpha|-\beta+m}.$$

The interest in the multiparameter resolvent trace expansions arose in view their application for the computation of regularized sums of zeta-determinants in a joint project with Matthias Lesch. In particular, consider a manifold with fibered boundary $\mathbb{S}^d \times M$, with an elliptic boundary problem $(\mathscr{A}, \mathscr{B})$ represented after the eigenspace decomposition on $\mathbb{S}^d$ by an infinite sum of elliptic boundary problems $(A_n, B_n), n \in \mathbb{N}_0$ on M. Assume, the boundary problems admit well defined zeta-determinants. Then the methods elaborated in [LeVe13] together with our main theorem above equate the zeta-determinant of $\mathscr{A}_{\mathscr{B}}$ to the regularized sum of zeta determinants for A_{n,B_n}, up to a locally computable error term. We refer to [LeVe13] for further reference.

This paper is organized as follows. In §2 we recast the symbolic expansion of the resolvent $(A_B + z^q)^{-1}$ in terms of polyhomogeneity properties of its Schwartz kernel lifted to an appropriate blowup of $\mathbb{R}^+ \times M^2$. In §3 we establish a composition result for the polyhomogeneous conormal distributions on the blowup of $\mathbb{R}^+ \times M^2$. In §4 we employ this microlocal characterization of the resolvent kernel to establish the multiparameter resolvent trace expansion.

Contents

2. Resolvent kernel as a polyhomogeneous conormal distribution

Seeley [See69] provides a careful construction of the resolvent for the elliptic boundary problem A_B. More precisely, in view of [See69, Theorem 1, Lemma 2, (25), (32), (49))] we may state the following

Theorem 2.1. *Let $\Gamma_R := \{\mu \in \Gamma \mid |\mu| \geq R\}$. Consider a local coordinate neighborhood $\mathscr{U} \subset \partial M \times [0,1)_x$ in the collar of the boundary, with local coordinates $\{x, y = (y_1, .., y_{m-1})\}$. Then for any $j \in \mathbb{N}_0$ there exist $d_{-q-j} \in C^\infty(\mathscr{U} \times \mathbb{R}^m \times \Gamma, \operatorname{Hom}(E \restriction \mathscr{U}))$, homogeneous of oder $(-q-j)$ in $(x^{-1}, \zeta, \gamma, \mu)$, where $(x, y, \zeta, \gamma, \mu) \in \mathscr{U} \times \mathbb{R}^{m-1}_\zeta \times \mathbb{R}_\gamma \times \Gamma_\mu$, such that for $R > 0$ sufficiently large*

and $\mu \in \Gamma_R$ the Schwartz kernel

$$\begin{aligned}&(A_B+\mu^q)^{-1}(x,y,\widetilde{x},\widetilde{y}) - (A+\mu^q)^{-1}(x,y,\widetilde{x},\widetilde{y})\\ &-\sum_{j=0}^{N-1}(2\pi)^{-m}\int_{\mathbb{R}^{m-1}}\int_{\mathbb{R}} e^{i\langle y-\widetilde{y},\zeta\rangle}e^{-i\widetilde{x}\gamma}d_{-q-j}(x,y,\zeta,\gamma,\mu)d\gamma d\zeta\\ &=: K_{A_B}(x,y,\widetilde{x},\widetilde{y};\mu) - K_A(x,y,\widetilde{x},\widetilde{y};\mu) - \sum_{j=0}^{N-1}\mathrm{Op}(d_{-q-j}),\end{aligned}$$

is uniformly $O(|\mu|^{m-q-N})$ as $|\mu|\to\infty$. Here, the first term K_{A_B} denotes the resolvent kernel of A_B near the boundary, whereas the second term K_A refers to the interior resolvent parametrix of A, defined without taking into account the boundary conditions.

Assume for simplicity that μ varies along a ray within Γ, which we identify with $\mathbb{R}^+$. Then the Schwartz kernel K_{A_B} of the resolvent $(A_B+\mu^q)^{-1}$ is a distribution on $\mathbb{R}^+_{1/\mu}\times M^2$. Choose local coordinates (x,y) and $(\widetilde{x},\widetilde{y})$ on the two copies of M in a collar neighborhood of the boundary, where x and $\widetilde{x}$ are the boundary defining functions, with non-uniform behaviour at

$$\begin{aligned}\mathscr{D} &:= \{\mu=\infty,(x,y)=(\widetilde{x},\widetilde{y})\},\\ \mathscr{C} &:= \{\mu=\infty, x=\widetilde{x}=0, y=\widetilde{y}\}.\end{aligned} \tag{2.1}$$

This non-uniform behaviour is resolved by considering an appropriate blowup $\mathscr{M}^2_b$ of $\mathbb{R}^+\times M^2$ at $\mathscr{C}$ and $\mathscr{D}$, a procedure introduced by Melrose, see [Mel93], such that the kernels K_{A_B}, K_A lift to polyhomogeneous distributions on the manifold with corners $\mathscr{M}^2_b$ in the sense of the following definition.

Definition 2.2. *Let X be a manifold with corners, with all boundary faces embedded, and $\{(H_i,\rho_i)\}_{i=1}^N$ an enumeration of its boundaries and the corresponding defining functions. For any multi-index $b=(b_1,\dots,b_N)\in\mathbb{C}^N$ we write $\rho^b=\rho_1^{b_1}\dots\rho_N^{b_N}$. Denote by $\mathcal{V}_b(X)$ the space of smooth vector fields on X which lie tangent to all boundary faces. All distributions on X are locally restrictions of distributions defined across the boundaries of X. A distribution ω on X is said to be conormal if $\omega\in\rho^bC^\infty(X)$ for some $b\in\mathbb{C}^N$, and $V_1\dots V_\ell\omega\in\rho^bC^\infty(X)$, for all $V_j\in\mathcal{V}_b(X)$ and for every $\ell\ge 0$. An index set $E_i=\{(\gamma,p)\}\subset\mathbb{C}\times\mathbb{N}$ satisfies the following hypotheses:*

1. $\mathrm{Re}(\gamma)$ *accumulates only at plus infinity,*
2. *For each γ there is $P_\gamma\in\mathbb{N}_0$, such that $(\gamma,p)\in E_i$ iff $p\le P_\gamma$,*
3. *If $(\gamma,p)\in E_i$, then $(\gamma+j,p')\in E_i$ for all $j\in\mathbb{N}$ and $0\le p'\le p$.*

An index family $E=(E_1,\dots,E_N)$ is an N-tuple of index sets. Finally, we say that a conormal distribution w is polyhomogeneous on X with index family E, we write $\omega\in\mathscr{A}^E_{\mathrm{phg}}(X)$, if ω is conormal and if in addition, near each H_i,

$$\omega\sim\sum_{(\gamma,p)\in E_i} a_{\gamma,p}\rho_i^\gamma(\log\rho_i)^p, \ \text{ as } \rho_i\to 0, \tag{2.2}$$

with coefficients $a_{\gamma,p}$ conormal on H_i, polyhomogeneous with index E_j at any $H_i\cap H_j$.

We also need to consider polyhomogeneous distributions on a manifold with corners X, conormal to an embedded submanifold $Y\subset X$. The basic space $I^m(\mathbb{R}^n,\{0\})$

consists of compactly supported distributions with the Fourier transform given by a symbol of order $(m - n/4)$. $I^m(\mathbb{R}^n, \{0\})$ is invariant under local diffeomorphisms and thus makes sense on any manifold around an isolated point.

For an embedded k-submanifold $S \subset X$, any point in S admits an open neighborhood $\mathscr{V}$ in X which can be locally decomposed as a product $\mathscr{V} = X' \times X''$ so that $\mathscr{V} \cap S = X' \times \{p\}, p \in X''$. The space $I^m(X,S)$ is defined (locally) as the space of smooth functions on X' with values $I^{m+\dim X'/4}(X'', \{p\})$. The normalization is chosen to give pseudo-differential operators their expected orders. All distributions in $I^m(X,S)$ are locally restrictions of distributions on an ambient space, which are conormal to any smooth extension of S across ∂X.

Choosing now index sets E for each boundary face of X as in Definition 2.2, we define a space $\mathscr{A}^E_{\text{phg}}(X,Y)$ as the space of distributions conormal to Y, with polyhomogeneous expansions as in Eq. (2.2) at all boundary faces and with coefficients conormal to the intersection of Y with each boundary face.

We now continue with the definition of a blowup $\mathscr{M}^2_b$, so that the Schwartz kernels K_A and $\text{Op}(d_{-q-j})$ lift to polyhomogeneous distributions, possibly conormal to an embedded submanifold. Blowing up $\mathbb{R}^+ \times M^2$ at $\mathscr{C}$ and $\mathscr{D}$ amounts in principle to introducing polar coordinates in $\mathbb{R}^+ \times \mathbb{R}^+_2$ at $\mathscr{C}$ and $\mathscr{D}$ together with a unique minimal differential structure with respect to which these coordinates are smooth.

We first perform a blowup of $\mathscr{C}$. The resulting space $[\mathbb{R}^+ \times M^2, \mathscr{C}]$ is defined as the union of $\mathbb{R}^+ \times M^2 \setminus \mathscr{C}$ with the interior spherical normal bundle of $\mathscr{C}$ in $\mathbb{R}^+ \times M^2$. The blowup $[\mathbb{R}^+ \times M^2, \mathscr{C}]$ is endowed with the unique minimal differential structure with respect to which smooth functions in the interior of $\mathbb{R}^+ \times M^2$ and polar coordinates on $\mathbb{R}^+ \times M^2$ around $\mathscr{C}$ are smooth. This blowup introduces a new boundary hypersurface, which we refer to as the front face ff. The other boundary faces are as follows. The right face rf is the lift of $\{x = 0\}$, the left face lf is the lift of $\{\widetilde{x} = 0\}$, and the temporal face tf is the lift of $\{\mu = \infty\}$.

The actual blowup space $\mathscr{M}_b$ is obtained by a blowup of $[\mathbb{R}^+ \times M^2, \mathscr{C}]$ along the lift of the diagonal $\mathscr{D}$. The resulting blowup space $\mathscr{M}_b$ is defined as before by cutting out the submanifold and replacing it with its spherical normal bundle. This second blowup introduces an additional boundary hypersurface td, the temporal diagonal. $\mathscr{M}_b$ is a manifold with boundaries and corners, illustrated below.

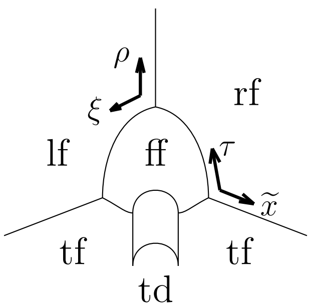

FIGURE 1. The blowup $\mathscr{M}^2_b = [[\mathbb{R}^+ \times M^2, \mathscr{C}], \mathscr{D}]$.

Denote by $Y := \{(\mu, p, \widetilde{p}) \in \mathbb{R}^+ \times M^2 \mid p = \widetilde{p}\}$ the diagonal hypersurface. We denote by $\mathscr{A}^{l,p}_{\text{phg}}(\mathscr{M}^2_b, \beta^* Y)$ the space of Schwartz kernels that lift to polyhomogeneous

conormal distributions on the blowup space $\mathscr{M}_b^2$, with leading order $(-m+l)$ at the front face ff, leading order $(-m+p)$ at the temporal diagonal td, index sets $(\mathbb{N}_0, \mathbb{N}_0)$ at the left and right boundary faces, conormal at the interior singularity β^*Y. The space of such Schwartz kernels without a conormal singularity is denoted by $\mathscr{A}^{l,p}_{\text{phg}}(\mathscr{M}_b^2)$.

Projective coordinates on $\mathscr{M}_b$ are given as follows. Near the top corner of ff away from tf the projective coordinates are given by

$$\rho = \frac{1}{\mu},\ \xi = \frac{x}{\rho},\ \widetilde{\xi} = \frac{\widetilde{x}}{\rho},\ w = \frac{y-\widetilde{y}}{\rho},\ y, \tag{2.3}$$

where in these coordinates $\rho, \xi, \widetilde{\xi}$ are the defining functions of the faces ff, rf and lf respectively. For the bottom corner of ff near rf the projective coordinates are given by

$$\tau = (\mu\widetilde{x})^{-1},\ s = \frac{x}{\widetilde{x}},\ u = \frac{y-\widetilde{y}}{\widetilde{x}},\ \widetilde{x},\ y, \tag{2.4}$$

where in these coordinates $\tau, s, \widetilde{x}$ are the defining functions of tf, rf and ff respectively. For the bottom corner of ff near lf the projective coordinates are obtained by interchanging the roles of x and $\widetilde{x}$. The projective coordinates on $\mathscr{M}_b$ near the top of td away from tf are given by

$$\eta = \tau,\ S = \frac{s-1}{\eta},\ U = \frac{u}{\eta},\ \widetilde{x}, \tag{2.5}$$

In these coordinates tf is the face in the limit $|(S,U)| \to \infty$, ff and td are defined by $\widetilde{x}, \eta$, respectively. The blowup $\mathscr{M}_b$ is related to the original space $\mathbb{R}^+ \times M^2$ via the obvious 'blow-down map'

$$\beta : \mathscr{M}_b \to \mathbb{R}^+ \times M^2,$$

which is in local coordinates simply the coordinate change back to $(1/\mu, (x,y), (\widetilde{x}, \widetilde{y}))$. The blowup $\mathscr{M}_b^2$ is similar to the blowup space construction for incomplete conical singularities by Mooers [Moo99] with the difference that here the blowup is not parabolic in $\mu^{-1}-$direction.

Theorem 2.3. *Consider the elliptic boundary value problem (A, B) with a differential operator A of order q on a compact manifold M of dimension m with boundary ∂M. Then the resolvent kernel K_{A_B} and the interior resolvent kernel K_A are both elements of $\mathscr{A}^{q,q}_{\text{phg}}(\mathscr{M}_b^2, \beta^*Y)$, where $Y := \{(\mu, p, \widetilde{p}) \in \mathbb{R}^+ \times M^2 \mid p = \widetilde{p}\}$ denotes the diagonal hypersurface. Moreover, their difference $K_{A_B} - K_A =: K \in \mathscr{A}^{q,\infty}_{\text{phg}}(\mathscr{M}_b^2)$.*

Proof. For $R > 0$ sufficiently large and $\mu \in \Gamma_R$, we may write according to [See69, (26), (28)]

$$\text{Op}(d_{-q-j}) = (2\pi)^{-m} \int_{\mathbb{R}^{m-1}} e^{i\langle y-\widetilde{y}, \zeta\rangle} \widetilde{d}_{-q-j}(x, y, \zeta, \widetilde{x}, \mu)\, d\zeta, \tag{2.6}$$

where $\widetilde{d}_{-q-j}$ is homogeneous of degree $(-q-1)$ in $(x^{-1}, \widetilde{x}^{-1}, \zeta, \mu)$. Moreover, [See69, (29)] asserts the following estimate

$$\begin{aligned} \left| x^i \widetilde{x}^k \partial_x^\alpha \partial_{\widetilde{x}}^\beta \partial_\zeta^\gamma \partial_\mu^\delta \widetilde{d}_{-q-j}(x, y, \zeta, \widetilde{x}, \mu) \right| &\leq C \exp\left(-c(x+\widetilde{x})(|\zeta|+\mu)\right) \\ &\quad \times (|\zeta|+\mu)^{1-q-j-k-i+\alpha+\beta-|\gamma|-\delta}, \end{aligned} \tag{2.7}$$

with constants $c, C > 0$. This estimate is stable under differentiation in $y \in \mathbb{R}^{m-1}$. We may now study the asymptotics of the lift $\beta^* \mathrm{Op}(d_{-q-j})$ in the various projective coordinates near the front face of $\mathscr{M}^2_b$. For instance, in coordinates (2.4) we find

$$\beta^* \mathrm{Op}(d_{-q-j}) = (2\pi)^{-m} \widetilde{x}^{-m+q+j} \int_{\mathbb{R}^{m-1}} e^{i\langle u, \nu\rangle} \widetilde{d}_{-q-j}(s, y, \nu, 1, \tau^{-1})\, d\nu. \tag{2.8}$$

Hence the lift $\beta^* \mathrm{Op}(d_{-q-j})$ is of order $(-m+q+j)$ at the front face ff ($\widetilde{x} \to 0$), and smooth at rf ($s \to 0$). In view of the estimate (2.7) the expression is also vanishing to infinite order at the temporal face tf ($\tau \to 0$). Similarly, in coordinates (2.5)

$$\beta^* \mathrm{Op}(d_{-q-j}) = (2\pi)^{-m} (\eta\widetilde{x})^{-m+q+j} \int_{\mathbb{R}^{m-1}} e^{i\langle U, \nu\rangle} \widetilde{d}_{-q-j}(S + \eta^{-1}, y, \nu, \eta^{-1}, 1)\, d\nu. \tag{2.9}$$

Hence the lift $\beta^* \mathrm{Op}(d_{-q-j})$ is of order $(-m+q+j)$ at the front face ff ($\widetilde{x} \to 0$). In view of the estimate (2.7) the expression is also vanishing to infinite order at the temporal face tf ($|(S, U)| \to \infty$) as well as temporal diagonal td ($\eta \to 0$).

Summarizing we have shown $\beta^* \mathrm{Op}(d_{-q-j}) \in \mathscr{A}^{q,\infty}_{\mathrm{phg}}(\mathscr{M}^2_b)$. Similar arguments applied to the classical symbol expansion of the interior parametrix $K_A = (A + \mu^q)^{-1}$ explain $K_A \in \mathscr{A}^{q,q}_{\mathrm{phg}}(\mathscr{M}^2_b, \beta^* Y)$. Since $\beta^*\mu = \rho_{\mathrm{ff}}\rho_{\mathrm{td}}\rho_{\mathrm{tf}}$, we infer from Theorem 2.1

$$\beta^* K_{A_B} = \beta^* K_A + \sum_{j=0}^{N-1} \beta^* \mathrm{Op}(d_{-q-j}) + O(\rho_{\mathrm{ff}}\rho_{\mathrm{td}}\rho_{\mathrm{tf}})^{-m+q+N}, \tag{2.10}$$

as $(\rho_{\mathrm{ff}}, \rho_{\mathrm{td}}, \rho_{\mathrm{tf}}) \to 0$. Taking the limit $N \to \infty$ proves at the statement. □

3. Composition of polyhomogeneous Schwartz kernels

Let X and X' be two compact manifolds with corners, and let $f : X \to X'$ be a smooth map. Let $\{H_i\}_{i\in I}$ and $\{H'_j\}_{j\in J}$ be enumerations of the codimension one boundary faces of X and X', respectively, and let ρ_i, ρ'_j be global defining functions for H_i, resp. H'_j. We say that the map f is a b-map if

$$f^*\rho'_j = A_{ij} \prod_{i\in I} \rho_j^{e(i,j)}, \quad A_{ij} > 0,\ e(i,j) \in \mathbb{N} \cup \{0\}.$$

The map f is called a b-submersion if f_* induces a surjective map between the b-tangent bundles of X and X'. The notion of b-tangent bundles has been introduced in [Mel93]. Assume moreover that for each j there is at most one i such that $e(i,j) \neq 0$. In other words no hypersurface in X gets mapped to a corner in X'. Under this condition the b-submersion f is called a b-fibration.

Suppose that ν_0 is a density on X which is smooth up to all boundary faces and everywhere nonvanishing. A smooth b-density ν_b is, by definition, any density of the form $\nu_b = \nu_0(\Pi\rho_i)^{-1}$. Let us fix smooth b-densities ν_b on X and ν'_b on X'.

Proposition 3.1. [Mel92, The Pushforward Theorem] *Let $f_b : X \to X'$ be a b–fibration. Let u be a polyhomogeneous function on X with index sets E_i the faces H_i of X. Suppose that each $(z,p) \in E_i$ has $Re\, z > 0$ if $e(i,j) = 0$ for all $j \in J$. Then the pushforward $f_*(u\nu_b)$ is well-defined and equals $h\nu'_b$ where h is polyhomogeneous on X' and has an index family $f_b(\mathcal{E})$ given by an explicit formula in terms of the index family $\mathcal{E}$ for X.*

Rather than giving the formula for the image index set in general, we provide the index image set in a specific setup, enough for the present situation. If H_{i_1} and H_{i_2} are both mapped to a face H'_j, and if $H_{i_1} \cap H_{i_2} = \emptyset$, then they contribute to the index set $E_{i_1} + E_{i_2}$ to H'_j. If they do intersect, however, then the contribution is the extended union $E_{i_1} \overline{\cup} E_{i_2}$

$$E_{i_1} \overline{\cup} E_{i_2} := E_{i_1} \cup E_{i_2} \cup \{((z, p+q+1) : \exists\, (z,p) \in E_{i_1}, \text{ and } (z,q) \in E_{i_2}\}.$$

We now employ the Pushforward theorem to establish the following fundamental composition result. Let $Y := \{(\mu, p, \widetilde{p}) \in \mathbb{R}^+ \times M^2 \mid p = \widetilde{p}\}$ denote the diagonal hypersurface. Consider any $K_a \in \mathscr{A}^{\ell,k}_{\mathrm{phg}}(\mathscr{M}^2_b, \beta^* Y)$ and $K_b \in \mathscr{A}^{\ell',\infty}_{\mathrm{phg}}(\mathscr{M}^2_b)$. Their composition is defined by

$$K_c(p, \widetilde{p}; \mu) = \int_M K_a(p, p'; \mu) K_b(p', \widetilde{p}; \mu) \, \mathrm{dvol}_M(p'). \tag{3.1}$$

Proposition 3.2.

$$\mathscr{A}^{\ell,k}_{\mathrm{phg}}(\mathscr{M}^2_b, \beta^* Y) \circ \mathscr{A}^{\ell',\infty}_{\mathrm{phg}}(\mathscr{M}^2_b) \subset \mathscr{A}^{\ell+\ell',\infty}_{\mathrm{phg}}(\mathscr{M}^2_b). \tag{3.2}$$

Proof. Consider $K_a \in \mathscr{A}^{l,k}_{\mathrm{phg}}(\mathscr{M}^2_b, \beta^* Y)$ and $K_b \in \mathscr{A}^{l',\infty}_{\mathrm{phg}}(\mathscr{M}^2_b)$. Their composition $K_c = K_a \circ K_b$ is defined in (3.1). This expression can be rephrased in microlocal terms. Consider the space $\mathbb{R}^+_{1/\mu} \times M^3_{(p,p',\widetilde{p})}$, and the three projections

$$\begin{aligned} \pi_c &: \mathbb{R}^+_{1/\mu} \times M^3_{(p,p',\widetilde{p})} \to \mathbb{R}^+_{1/\mu} \times M^2_{(p,\widetilde{p})}, \\ \pi_a &: \mathbb{R}^+_{1/\mu} \times M^3_{(p,p',\widetilde{p})} \to \mathbb{R}^+_{1/\mu} \times M^2_{(p,p')}, \\ \pi_b &: \mathbb{R}^+_{1/\mu} \times M^3_{(p,p',\widetilde{p})} \to \mathbb{R}^+_{1/\mu} \times M^2_{(p',\widetilde{p})}. \end{aligned} \tag{3.3}$$

We reinterpret K_a, K_b and K_c as 'right densities'

$$\begin{aligned} K_a &\equiv K_a(p, p'; \mu) \, \mathrm{dvol}_M(p'), \\ K_b &\equiv K_b(p', \widetilde{p}; \mu) \, \mathrm{dvol}_M(\widetilde{p}), \\ K_c &\equiv K_c(p, \widetilde{p}; \mu) \mathrm{dvol}_M(\widetilde{p}). \end{aligned}$$

Then we can rewrite (3.1) as

$$K_c = (\pi_c)_* \left(\pi_a^* K_a \cdot \pi_b^* K_c\right).$$

The basic idea in the proof of polyhomogeneity of K_c is a construction of a triple-space $\mathscr{M}^3_b$ which is a blowup of $\mathbb{R}^+_{1/\mu} \times M^3$ obtained by a sequence of blowups, designed such that there are maps

$$\Pi_a, \Pi_c, \Pi_b : \mathscr{M}^3_b \longrightarrow [\mathbb{R}^+_{1/\mu} \times M^2, \mathscr{C}] =: \mathscr{M}^2_{rb}$$

which 'cover' the three projections defined above. The construction is reminiscent of the triple space construction for the heat space calculus for conical singularities, see [Moo99], but differs from the latter since there is no convolution in the parameter μ^{-1} variable and the blowups are not parabolic in the μ^{-1} direction. On each copy of M we use the local coordinates $p = (x, y), p' = (x', y'), \widetilde{p} = (\widetilde{x}, \widetilde{y}) \in \mathbb{R}^+ \times \mathbb{R}^{m-1}$ near the boundary ∂M with $(x, x', \widetilde{x})$ being the three copies of the boundary defining function. First we blow up the submanifold

$$\begin{aligned} F &= \{(x, y, x', y', \widetilde{x}, \widetilde{y}, \mu) \mid \mu = \infty, x = x' = \widetilde{x} = 0, y = y' = \widetilde{y}\} \\ &= \pi_a^{-1}\mathscr{C} \cap \pi_b^{-1}\mathscr{C} \cap \pi_c^{-1}\mathscr{C}, \end{aligned}$$

which is the intersection of all highest codimension corners $\mathscr{C}$ introduced in (2.1), pulled back under the three projections to $\mathbb{R}^+ \times M^3$. Then we blow up the resulting space $[\mathbb{R}^+ \times M^3, F]$ at the lifts of each of the three submanifolds

$$\begin{aligned} F_c &= \pi_c^{-1}\mathscr{C} = \{(x, y, x', y', \widetilde{x}, \widetilde{y}, \mu) \mid \mu = \infty, x = \widetilde{x} = 0, y = \widetilde{y}\}, \\ F_a &= \pi_a^{-1}\mathscr{C} = \{(x, y, x', y', \widetilde{x}, \widetilde{y}, \mu) \mid \mu = \infty, x = x' = 0, y = y'\}, \\ F_b &= \pi_b^{-1}\mathscr{C} = \{(x, y, x', y', \widetilde{x}, \widetilde{y}, \mu) \mid \mu = \infty, x' = \widetilde{x} = 0, y' = \widetilde{y}\}. \end{aligned} \tag{3.4}$$

Identifying notationally each $F_{a,b,c}$ with their lifts to $[\mathbb{R}^+ \times M^3, F]$, we may altogether define the triple space

$$\mathscr{M}_b^3 := \left[\left[\mathbb{R}^+ \times M^3, F\right] F_a, F_b, F_c \right].$$

If we ignore the $\mu^{-1}-$direction, the spacial part of $\mathscr{M}_b^3$ can be visualized as below.

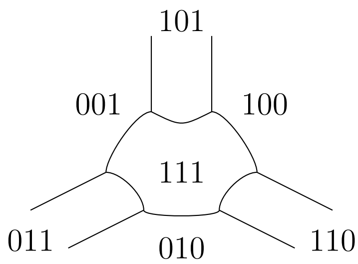

FIGURE 2. The spacial component of the triple space $\mathscr{M}_b^3$.

Here, $(101), (011)$ and (110) label the boundary faces created by blowing up F_c, F_b and F_a, respectively. The face (111) is the front face introduced by blowing up F. We denote the defining function for the face (ijk) by ρ_{ijk}. The triple space comes with a natural blowdown map $\beta^{(3)} : \mathscr{M}_b^3 \to \mathbb{R}^+ \times M^3$, which as in the discussion of $\mathscr{M}_b^2$ amounts in local coordinates to a coordinate change back to $(x, y, x', y', \widetilde{x}, \widetilde{y}, \mu)$.

Now consider the projections π_c, π_a and π_b introduced in (3.3). These induce projections Π_c, Π_a and Π_b from $\mathscr{M}_b^3$ to the reduced blowup space $\mathscr{M}_{rb}^2$. It is not hard to check that the choice of submanifolds $F, F_{a,b,c}$ that have been blown up ensures that these projections are in fact b-fibrations.

Denote the defining functions for the right, front and left faces of each copy of $\mathscr{M}_{rb}^2$ by $\{\rho_{10}, \rho_{11}, \rho_{01}\}$, respectively. These lift via the projections according to the following rules

$$\begin{aligned} \Pi_c^*(\rho_{ij}) &= \rho_{i0j}\rho_{i1j}, \\ \Pi_a^*(\rho_{ij}) &= \rho_{ij0}\rho_{ij1}, \\ \Pi_b^*(\rho_{ij}) &= \rho_{0ij}\rho_{1ij}. \end{aligned} \tag{3.5}$$

Now consider the behaviour in the parameter μ^{-1}-direction. Let τ be the defining function for the boundary face in $\mathscr{M}_b^3$ which is mapped onto $\{\mu = \infty\}$ by the blowdown map. Since each $F, F_{a,b,c}$ is a submanifold of $\{\mu = \infty\}$, we find

$$\beta_{(3)}^* \mu^{-1} = \tau \rho_{111} \rho_{110} \rho_{101} \rho_{011}. \tag{3.6}$$

Let $\beta_{(2)} : \mathscr{M}^2_{rb} \to \mathbb{R}^+ \times M^2$ be the blowdown map for the reduced blowup space. Then $\beta^*_{(2)}\mu^{-1} = T\rho_{11}$, where T is the defining function for the temporal face tf in $\mathscr{M}^2_{rb}$. Note that $\beta_{(2)} \circ \Pi_{a,b,c} = \pi_{a,b,c} \circ \beta_{(3)}$ and hence acting on functions on $\mathbb{R}^+ \times M^2$ we have

$$\Pi^*_{a,b,c} \circ \beta^*_{(2)} = \beta^*_{(3)} \circ \pi^*_{a,b,c}.$$

Consequently, in view of Eq. (3.5) and Eq. (3.6), we conclude

$$\begin{aligned} \Pi^*_c(T) &= \tau\rho_{110}\rho_{011}, \\ \Pi^*_a(T) &= \tau\rho_{101}\rho_{011}, \\ \Pi^*_b(T) &= \tau\rho_{101}\rho_{110}. \end{aligned} \tag{3.7}$$

Using these data, we may now prove the anticipated composition formula. Consider the 'right densities', $K_a(x,y,x',y';\mu)dx'dy'$ and $K_b(x',y',\widetilde{x},\widetilde{y};\mu)d\widetilde{x}d\widetilde{y}$. Their product is given by

$$K_a(x,y,x',y';\mu) \cdot K_b(x',y',\widetilde{x},\widetilde{y};\mu)\, dx'\, dy'\, d\widetilde{x}\, d\widetilde{y}$$

Its integral over $dx'dy'$ gives $K_c(x,y,\widetilde{x},\widetilde{y};\mu)d\widetilde{x}\,d\widetilde{y}$. To put this into the same form required in the pushforward theorem, write $t = \mu^{-1}$ and multiply this expression by $dt\,dx\,dy$.

Blowing up a submanifold of codimension n amounts in local coordinates to introducing polar coordinates, so that the coordinate transformation of a density leads to $(n-1)^{\text{st}}$ power of the radial function, which is the defining function of the corresponding front face. Hence we compute the lift

$$\begin{aligned} &\beta^*_{(3)}(dt\,dx\,dy\,dx'\,dy'\,d\widetilde{x}\,d\widetilde{y}) \\ &= \rho_{111}^{3+2(m-1)}\rho_{101}^{2+(m-1)}\rho_{110}^{2+(m-1)}\rho_{011}^{2+(m-1)}\nu^{(3)} \\ &= \rho_{111}^{3+2(m-1)}\rho_{101}^{2+(m-1)}\rho_{110}^{2+(m-1)}\rho_{011}^{2+(m-1)}\tau\,(\Pi\rho_{ijk})\,\nu_b^{(3)}, \end{aligned} \tag{3.8}$$

where $\nu^{(3)}$ is a density on $\mathscr{M}^3_b$, smooth up to all boundary faces and everywhere nonvanishing; $\nu_b^{(3)}$ is a b-density, obtained from $\nu^{(3)}$ by dividing by a product of all defining functions on $\mathscr{M}^3_b$; and $(\Pi\rho_{ijk})$ is a product over all $(ijk) \in \{0,1\}^3$. Set $\kappa_a = \beta^*_{(2)}K_a$ and $\kappa_b = \beta^*_{(2)}K_b$. Since κ_b is vanishing to infinite order as $T \to 0$, its lift $\Pi^*_b\kappa_b$ vanishes to infinite order in $\tau\rho_{110}\rho_{101}$ by Eq. (3.7).

Since κ_a is not polyhomogeneous on the reduced blowup space $\mathscr{M}^2_{rb}$, the lift $\Pi^*_a\kappa_a$ is not polyhomogeneous in τ. However, due to infinite order vanishing of $\Pi^*_b\kappa_b$ in τ, their product $\Pi^*_a\kappa_a \cdot \Pi^*_b\kappa_b$ is polyhomogeneous and vanishing to infinite order in $\tau\rho_{110}\rho_{101}\rho_{011}$. We obtain

$$\Pi^*_a\kappa_a \cdot \Pi^*_b\kappa_b\, \beta^*_{(3)}(dt\,dx\,dy\,dx'\,dy'\,d\widetilde{x}\,d\widetilde{y}) = \rho_{111}^{\ell+\ell'+1}\,(\Pi\rho_{ijk})\,G\nu_b^{(3)},$$

where G is a bounded polyhomogeneous function on $\mathscr{M}^3_b$, vanishing to infinite order in $(\tau\rho_{110}\rho_{101}\rho_{011})$, with index sets $\mathbb{N}_0$ at the faces (001), (100) and (010). Applying the Pushforward Theorem now gives

$$\begin{aligned} &(\Pi_c)_*\left(\Pi^*_a\kappa_a \cdot \Pi^*_b\kappa_b\, \beta^*_{(3)}(dt\,dx\,dy\,dx'\,dy'\,d\widetilde{x}\,d\widetilde{y})\right) \\ &= \beta^*_{(2)}\left(K_c\,dt\,dx\,dy\,d\widetilde{x}\,d\widetilde{y}\right) = \rho_{11}^{2+\ell+\ell'}\,G'\,\nu_b^{(2)}, \end{aligned} \tag{3.9}$$

where $\nu_b^{(2)}$ is a b-density on $\mathscr{M}^2_{rb}$ and G' is a bounded polyhomogeneous function on $\mathscr{M}^2_{rb}$, which vanishes to infinite order in T, and has the index set $\mathbb{N}_0$ at the left and

right boundary faces. By [EMM91, Proposition B7.20] the pushforward is smooth across β^*Y.

Note also that the pushforward by Π_c does not introduce logarithmic terms in the front face expansion of κ_c, since the kernel on $\mathscr{M}^2_{rb}$ is vanishing to infinite order at (101). Hence, for κ_A and κ_B with integer exponents in their front face expansions, same holds for their composition.

By an argument similar to Eq. (3.8), we compute

$$\beta^*_{(2)}(dt\, dx\, dy\, d\widetilde{x}\, d\widetilde{y}) = \rho_{11}^{m+1}\, (\rho_{10}\rho_{11}\rho_{01}T)\, \nu_b^{(2)}. \tag{3.10}$$

Consequently, combining Eq. (3.9) and Eq. (3.10), we deduce that $\beta^*_{(2)}K_c$ vanishes to infinite order in T, is of leading order $(-m+\ell+\ell')$ at the front face and has the index sets $\mathbb{N}_0$ at the left and right boundary faces. This proves the statement. $\square$

4. Multiparameter resolvent trace expansion

We continue in the notation introduced in §1. We abbreviate

$$\mu^q := \sum_{k=1}^n \lambda_k^q V_k(0) + z^q, \quad \lambda(V,W) := \sum_{k=1}^n \lambda_k^q (V_k - V_k(0)) + W,$$

and expand $A_B(\lambda)^{-1}$ in Neumann series, for the moment only formally

$$\begin{aligned}(A_B(\lambda)+z^q)^{-1} &=: \left(I + (A_B+\mu^q)^{-1}\lambda(V,W)\right)(A_B+\mu^q)^{-1} \\ &= \sum_{j=0}^\infty (-1)^j \left((K_{A_B}\lambda(V,W))^j K_{A_B} - (K_A\lambda(V,W))^j K_A \right) \\ &+ \sum_{j=0}^\infty (-1)^j (K_A\lambda(V,W))^j K_A =: \sum_{j=0}^\infty (-1)^j R_0^j + R_1. \\ &=: R_0 + R_1.\end{aligned} \tag{4.1}$$

The Neumann series converges in the operator norm for $z\in\Gamma$ with $|z|\gg 0$ sufficiently large. Trace norm estimates of R_0^j depend on the following basic result.

Proposition 4.1. *Consider $R \in \mathscr{A}^{\ell,\infty}_{\mathrm{phg}}(\mathscr{M}^2_b)$. Then R defines a Hilbert Schmidt operator with the Hilbert Schmidt norm*

$$\|R(\cdot,\mu)\|_{HS} = O(\mu^{-\ell+m/2}),\ \mu\to\infty.$$

Proof. We explify the argument in projective coordinates (2.4). R lifts to a polyhomogeneous function on $\mathscr{M}^2_{rb}$ and

$$\begin{aligned}\beta^*(R^2 dx\, dy\, d\widetilde{x}\, d\widetilde{y}) &= \widetilde{x}^{-m+2\ell} G\, ds\, d\widetilde{x}\, du\, dy \\ &= \mu^{-2\ell+m}\tau^{m-2\ell} G\, ds\, d\widetilde{x}\, du\, dy,\end{aligned}$$

where G is bounded in $(\widetilde{x},s,u,y)$ and vanishing to infinite order as $\tau\to\infty$. Similar estimates hold at the other two corners of the front face in the reduced blowup space $\mathscr{M}^2_{rb}$. Consequently we obtain as $\mu\to\infty$

$$\|R(\cdot,\mu)\|^2_{HS} = \int_M\int_M R(p,\widetilde{p};\mu)^2 \mathrm{dvol}_M(p)\mathrm{dvol}_M(\widetilde{p}) = O(\mu^{-2\ell+m}).$$

$\square$

Corollary 4.2. *Let $M \in \mathbb{N}$ and $\beta \in \mathbb{N}_0$ be fixed. Write for any multiindex $\alpha \in \mathbb{N}_0^n$ $\partial_\lambda^\alpha = \partial_{\lambda_1}^{\alpha_1} \cdots \partial_{\lambda_n}^{\alpha_n}$. Then for μ_0 sufficiently large there exist constants $C > 0$ and $0 < q < 1$ such that for $N \geq M$ and $\mu \geq \mu_0$ the trace norms satisfy the following estimates*

$$\begin{aligned} &\left\| \partial_\lambda^\alpha \partial_z^\beta R_0^N \right\|_{\mathrm{tr}} \leq C \cdot N \cdot q^{N-M} \cdot \mu^{-M-|\alpha|-\beta-q+\frac{m}{2}}, \\ &\left\| \partial_\lambda^\alpha \partial_z^\beta \sum_{j=M}^\infty R_0^j \right\|_{\mathrm{tr}} = O\big(\mu^{-M-|\alpha|-\beta-q+\frac{m}{2}}\big), \text{ as } \mu \to \infty. \end{aligned} \tag{4.2}$$

Proof. It suffices to consider the case $\alpha = 0^n$ and $\beta = 0$. The case of general α, β follows easily by the following relations

$$\begin{aligned} &\partial_z \big(A_B + \mu^q\big)^{-1} = -q\, z^{q-1} \big(A_B + \mu^q\big)^{-2}, \\ &\partial_{\lambda_k} \big(A_B + \mu^2\big)^{-1} = -q\, \lambda_k^{q-1}\, V_k(0)\, \big(A_B + \mu^q\big)^{-2}, \end{aligned}$$

and similarly for the other involved Schwartz kernels. For elements a and b in a not necessarily commutative ring we have by induction the following identity

$$(a+b)^N - b^N = \sum_{k=0}^{N-1} b^k a (a+b)^{N-k-1}. \tag{4.3}$$

Consequently, we may write ($K := K_{A_B} - K_A$ introduced in Theorem 2.3)

$$\begin{aligned} R_0^N &= \left(K_{A_B} \lambda(V,W)\right)^N K_{A_B} - \left(K_A \lambda(V,W)\right)^N K_A \\ &= \sum_{k=0}^{N-1} \left(K_A \lambda(V,W)\right)^k \circ K\, \lambda(V,W) \circ \left(K_{A_B} \lambda(V,W)\right)^{N-k-1} \\ &+ \left(K_{A_B} \lambda(V,W)\right)^N \circ K. \end{aligned} \tag{4.4}$$

Each of the N summands of R_0^N is of the form

$$P_N = \left(\prod_{j=1}^N K_j \lambda(V,W) \right) K_0, \tag{4.5}$$

where each $K_j, j = 0, \ldots, N$, is either K_{A_B}, K_A or K. By construction, at least one of the kernels equals K.

The analysis of the resolvent asymptotics separates into the discussion of the interior and the boundary parametrix. In the interior our main result is a consequence of the strong parametric elliptic calculus, cf. [LeVe13]. Hence we may assume without loss of generality that after eventual multiplication with a cutoff function, $\mathrm{supp}(V_k) \subset [0, \delta)$ with $\delta > 0$ sufficiently small such that $\|V_k - V_k(0)\|_\infty \leq V_k(0)/2$ for each $k = 1, .., n$. By the classical resolvent decay, we then have for each j an estimate of the operator norm

$$\|K_j \lambda(V,W)\| \leq \frac{\sum_{k=1}^n \lambda_k^q \, \|V_k - V_k(0)\|_\infty + \|W\|_\infty}{\sum_{k=1}^n \lambda_k^q \, V_k(0) + z^2} \leq q, \tag{4.6}$$

for $\mu \geq \mu_0$ sufficiently large and some $0 < q < 1$. Thus we may estimate the trace norm of each summand P_N by

$$\left\| P_N \right\|_{\mathrm{tr}} \leq q^{N-M} \cdot \Big\| \left(\prod_{j=1}^M K_j \lambda(V,W) \right) K_0 \Big\|_{\mathrm{HS}}, \tag{4.7}$$

where $\|\cdot\|_{\mathrm{tr}}, \|\cdot\|_{\mathrm{HS}}$ denote the trace norm resp. the Hilbert-Schmidt norm. At least one of K_j's in (4.5) equals K, and by choosing those factors whose norm we estimate by q appropriately we can arrange that in Eq. (4.7) at least one of the $K_j, j = 0, \dots, M$ equals K. Hence

$$\left(\prod_{j=1}^{M} K_j \lambda(V,W)\right) K_0 \in \bigoplus_{p=0}^{M} \bigoplus_{k=1}^{n} \lambda_k^{qp} \mathscr{A}_{\mathrm{phg}}^{q(M+1)+p,\infty}(\mathscr{M}_b^2). \tag{4.8}$$

By Proposition 4.1 we find

$$\|\left(\prod_{j=1}^{M} K_j \lambda(V,W)\right) K_0\|_{\mathrm{HS}} \leq C_M \, \mu^{-M-q+m/2}. \tag{4.9}$$

This proves the first and subsequently the second statements. □

We may now separate

$$R_0 = \sum_{j=0}^{\infty} (-1)^j R_0^j = \sum_{j=0}^{M-1} (-1)^j R_0^j + \sum_{j=M}^{\infty} (-1)^j R_0^j,$$

and in view of Corollary 4.2, it suffices to discuss the first summand only, consisting only of finitely many summands. The trace expansion of these follows by the next fundamental result.

Proposition 4.3. *For any $R \in \mathscr{A}_{\mathrm{phg}}^{\ell,\infty}(\mathscr{M}_b^2)$ we find*

$$\mathrm{Tr}\, R(\cdot, \mu) \sim \sum_{j=0}^{\infty} a_j \mu^{-1-j-\ell+m}, \ \ \mu \to \infty. \tag{4.10}$$

Proof. Consider $Y := \{(\mu, p, \widetilde{p}) \in \mathbb{R}^+ \times M^2 \mid p = \widetilde{p}\}$. The lift $\beta^* R$ does not have a conormal singularity and restricts to a polyhomogeneous distribution on $\beta^* Y \subset \mathscr{M}_b^2$, which itself is a blowup of $\mathbb{R}^+_{1/\mu} \times M$ at the highest codimension corner, with the blowdown map denoted by β_Y. We refer to the restrictions of ff, lf and td in $\mathscr{M}_b^2$ to $\beta^* Y$ as the front face, left face and temporal diagonal again.

The restriction of $\beta^* R$ to $\beta^* Y$ is polyhomogeneous of leading order $(-m+\ell)$ at the front face, index set $\mathbb{N}_0$ at the left face and vanishes to infinite order at the temporal diagonal. Consider the obvious projection $\pi : \mathbb{R}^+ \times M \to \mathbb{R}^+$. Then $(t = 1/\mu)$

$$\mathrm{Tr}\, R(\cdot, \mu)\, dt = (\pi \circ \beta_Y)_* \left(\beta_Y^* R|_Y dt \, \mathrm{dvol}_M\right).$$

Note that

$$\beta_Y^* \left(R|_Y dt \, \mathrm{dvol}_M\right) = \rho_{\mathrm{ff}}^{-m+\ell+2} G \nu_b.$$

where ν_b is a b-density on $\beta^* Y$, ρ_{ff} is the defining function of the front face, G is a bounded polyhomogeneous distribution with index set $\mathbb{N}_0$ at the left face and vanishes to infinite order at the temporal diagonal in $\beta^* Y$. By the Theorem 3.1 we find as $t \to 0$

$$(\pi \circ \beta_Y)_* \left(\beta_Y^* R|_Y dt \, \mathrm{dvol}_M\right) \sim \sum_{j=0}^{\infty} t^{-m+\ell+2+j} \left(t^{-1} dt\right).$$

This proves the statement. □

Corollary 4.4. *Consider any multiindex $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0$. For each $i \in \mathbb{N}_0$ there exist $h_i \in C^\infty(\Gamma^{n+1}\backslash\{0\})$, homogeneous in (λ, z) of order $(-1-q+m-i)$, such that in the notation of Corollary 4.2*

$$\partial_\lambda^\alpha \partial_z^\beta \operatorname{Tr} R_0 \sim \sum_{i=0}^\infty h_{i+|\alpha|+\beta}(\lambda, z), \text{ as } \mu \to \infty.$$

Proof. As before in Corollary 4.2 it suffices to consider $\alpha = 0^n, \beta = 0$. By construction we may set

$$R_0^j = \sum_{p=0}^n \sum_{k=1}^n Q_{p,k}^j \in \bigoplus_{p=0}^j \bigoplus_{k=1}^n \lambda_k^{qp} \mathscr{A}_{\mathrm{phg}}^{q(j+1)+p,\infty}(\mathscr{M}_b^2).$$

For each $Q_{p,k}^j \in \lambda_k^{qp} \mathscr{A}_{\mathrm{phg}}^{q(j+1)+p,\infty}(\mathscr{M}_b^2)$ we find by Proposition 4.3 as $\mu \to \infty$

$$\operatorname{Tr} Q_{p,k}^j(\cdot, \mu) \sim \sum_{i=0}^\infty a_i \lambda_k^{qp} \mu^{-1-i-q(j+1)-p+m} \sim \sum_{i=0}^\infty b_i \, (\lambda_k/\mu)^{qp} \, \mu^{-1-i-j-q+m}.$$

Altogether we obtain the following polyhomogeneous multiparameter trace expansion for any $M \in \mathbb{N}$

$$\operatorname{Tr} \sum_{j=0}^{M-1} (-1)^j R_0^j \sim \sum_{i=0}^\infty h_i'(\lambda, z), \text{ as } \mu \to \infty,$$

where each $h_i'(\lambda, z)$ is homogeneous in (λ, z) of order $(-1-q+m-i)$. Taking $M \to \infty$ we derive by Corollary 4.2 the stated full polyhomogeneous expansion. □

It remains to study the polyhomogeneous multiparameter expansion of R_1 from Eq. (4.1), which is the interior parametrix to $(A_B(\lambda) + z^q)$. The multiparameter expansion of R_1 then follows from the calculus of pseudo-differential operators with parameter, for a survey type exposition see for example [Les10, Sec. 4 and 5].

The differential expression $(A(\lambda) + z^q)$ is of order q, elliptic in the parametric sense with parameter $(\lambda, z) \in \Gamma^{n+1}$. We write $(A(\lambda) + z^q) \in \mathrm{CL}^q(M; \Gamma^{n+1})$. By [Shu01, Sec. II.9], its parametrix $R_1 \in \mathrm{CL}^{-q}(M; \Gamma^{n+1})$, and the N-th power $R_1^N \in \mathrm{CL}^{-qN}(M; \Gamma^{n+1})$. Consider $N \in \mathbb{N}$, such that $qN > m$. Fix any multiindex $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0$. Then the Schwartz kernel of R_1^N, which we do not distinguish notationally from the operator, is continuous with an asymptotic expansion on the diagonal as $|(\lambda, z)| \to \infty$, $(\lambda, z) \in \Gamma^{n+1}$

$$\partial_\lambda^\alpha \partial_z^\beta R_1^N(p, p; \lambda, z) \sim \sum_{j=0}^\infty f_j \left(p, \frac{(\lambda, z)}{|(\lambda, z)|} \right) |(\lambda, z)|^{-qN-j-|\alpha|-\beta+m}, \tag{4.11}$$

see [Les10, Theorem 5.1]. The functions f_j are smooth on $M \times (\Gamma^{n+1} \cap \mathbb{S}^n)$ and the expansion Eq. (4.11) is uniform over the compact manifold M. This leads to our final main result.

Theorem 4.5. *Consider any multiindex $\alpha \in \mathbb{N}_0^n$ and $\beta \in \mathbb{N}_0$. Fix $N \in \mathbb{N}$ such that $qN > m$. Then there exist $e_i \in C^\infty(M \times (\Gamma^{n+1} \cap \mathbb{S}^n))$, such that*

$$\partial_\lambda^\alpha \partial_z^\beta \operatorname{Tr}(A_B(\lambda) + z^q)^{-N} \sim \sum_{j=0}^\infty e_j \left(\frac{(\lambda, z)}{|(\lambda, z)|} \right) |(\lambda, z)|^{-1-qN-j-|\alpha|-\beta+m},$$

where e_0 is purely a boundary contribution.

Proof. The statement follows from Corollary 4.4, Eq. (4.11) and

$$\begin{aligned}\partial_\lambda^\alpha \partial_z^\beta (A_B(\lambda)+z^q)^{-N} &= \partial_\lambda^\alpha \partial_z^\beta \left(-\frac{1}{q} z^{1-q}\partial_z\right)^{N-1} (A_B(\lambda)+z^q)^{-1} \\ &= \partial_\lambda^\alpha \partial_z^\beta \left(-\frac{1}{q} z^{1-q}\partial_z\right)^{N-1} R_0 + \partial_\lambda^\alpha \partial_z^\beta R_1^N.\end{aligned} \tag{4.12}$$

□

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